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THE SOBOLEV REGULARITY OF SOLUTIONS OF FIRST ORDER NONLINEAR EQUATIONS

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Abstract. In order to study the propagation of singularities for solutions to second order quasilinear strictly hyperbolic equations with boundary, we have to consider the regularity of solutions of first order nonlinear equations satisfied by a characteristic hypersurface. In this paper, we study the regularity compositions of the form $v(\varphi(x), x)$ with v and φ assumed to have limited Sobolev regularities and we use it to prove the regularity of solutions of the first order nonlinear equations.

1. Introduction

Let $u(t, x_1, x') \in H^s_{loc}(\mathbf{R} \times \mathbf{R}^n_+)$, where $\mathbf{R}^n_+ = \{(x_1, x') : x_1 > 0\}$, be a solution of the quasilinear equation

$$P_2(t, x, u, Du, D)u = f(t, x, u, Du), \tag{1.1}$$

where

$$P_2(t, x, u, Du, D) \equiv (\partial_t^2 - \sum_{\substack{(i,j)\neq(0,0)\\ i,j=0}}^{n} a_{ij}(t, x, u, Du)\partial_{x_i}\partial_{x_j}) \tag{1.2}$$

be strictly hyperbolic with respect to $\{t = \mathit{constant}\}$. Suppose that a characteristic hypersurface for P_2 is given by $\{t = \varphi(x_1, x')\}$, where $x' = (x_2, \cdots, x_n) \in \mathbf{R}^{n-1}$. Then φ satisfies the first order nonlinear equation

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$$1 + 2(a_{01} + \sum_{j=2}^{n} a_{0j}\varphi_{x_j}) = a_{11}(\varphi_{x_1})^2 + 2\sum_{j=2}^{n} a_{1j}\varphi_{x_1}\varphi_{x_j}$$

$$(1.3) \qquad + \sum_{j=2}^{n} a_{jj}(\varphi_{x_j})^2 + \sum_{i,j=2}^{n} a_{ij}\varphi_{x_i}\varphi_{x_j}.$$

with coefficients evaluated at $t = \varphi(x_1, x')$. In order to consider conormal regularity of solutions u of the quasilinear equation $P_2 u = f(Du)$, where f is in the Sobolev space with suitable regularity, we have to study the regularity of $\varphi(x_1, x')$ satisfying the equation (1.3).

In this paper, we first examine the regularity of functions of the form $a(Du(\varphi(x_1, x'), x_1, x'))$. Then prove the regularity of solution $\varphi(x_1, x')$ satisfying the first order nonlinear equation (1.3) and the regularity of solution $\psi(x_1, x')$ satisfying the first order nonlinear equation $\psi_{x_1} = F_1 \tilde{D}\psi + F_2\psi + F_3$, where F_1 is in the space $H^{s-1}_{loc}(\mathbf{R}^n_+))$ and F_2, F_3 are in the space $H^{s-2}_{loc}(\mathbf{R}^n_+))$, $s > \frac{n}{2} + 2$.

2. Regularity of solutions of first order nonlinear equations

Before we prove the regularity of functions, we need the following well-known lemmas to prove the regularity of solutions of a first order nonlinear equation.

SCHAUDER'S LEMMA. *If $u, v \in H^s(\mathbf{R}^n)$ and $s > \frac{n}{2}$, then $uv \in H^s(\mathbf{R}^n)$ and $\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$.*

GAGLIARDO-NIRENBERG INEQUALITIES. *Let $1 \le q, r \le \infty$, and let m be an integer ≥ 2. If $u \in L^q(\mathbf{R}^n)$ and $D^\alpha u \in L^r(\mathbf{R}^n)$ when $|\alpha| = m$, then $D^\alpha u \in L^{p(\alpha)}(\mathbf{R}^n)$ for $|\alpha| \le m$, if*

$$m/p(\alpha) = (m - |\alpha|)/q + |\alpha|/r;$$

moreover, if $\|\cdot\|_s$ denotes the L^s norm, then

$$\sup_{|\alpha|=j} \|D^\alpha u\|_{p(\alpha)} \le 4^{|\alpha|(m-|\alpha|)}\big(\sup_{|\alpha|=m} \|D^\alpha u\|_r\big)^{|\alpha|/m}\|u\|_q^{(m-|\alpha|)/m}.$$

Throughout this paper we treat the case in which the regularity indices s and s' are integers; analogous results hold in the general case.

LEMMA 2.1. *Let* $0 \le s' \le s$ *and suppose that* $w \in L^\infty(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$. *Then for* $|\alpha| = s'$, *it follows that* $D^\alpha w \in L^{2p}(\mathbf{R}^n)$ *and*

$$\|D^\alpha w\|_{L^{2p}} \le C(\|w\|_{L^\infty})^{1-\frac{1}{p}}(\|w\|_{H^s})^{\frac{1}{p}},$$

where $p = s/s'$ *and* C *is a constant depending only on* s, s' *and* n.

We begin by studying the regularity of compositions of the form $v(\varphi(x'), x')$ with v and φ assumed to have limited Sobolev regularity. Even if $\varphi(x')$ is smooth, functions of the form $v(\varphi(x'), x')$ will in general have Sobolev regularity of order $1/2$ lower than that of $v(t, x')$. When $\varphi(x')$ is nonsmooth, the regularity of $v(\varphi(x'), x')$ will not in general be greater than that of $\varphi(x')$. In order to obtain norm estimates on the regularity of $v(\varphi(x'), x')$ that are linear in the norm of $\varphi(x')$, we will assume that the Sobolev regularity of $v(t, x')$ is at least one order greater than that of $\varphi(x')$.

LEMMA 2.2. *Let* $v(t, x_1, x') \in H^{s+1}_{loc}(\mathbf{R} \times \mathbf{R}^n_+)$ *for* $s > \frac{n}{2} + 1$. *Suppose that* $\varphi(x_1, x') \in H^{s'}_{loc}(\mathbf{R}^n_+)$, $1 \le s' \le s$, *and that* $D\varphi(x_1, x') \in L^\infty(\mathbf{R}^n_+)$. *Then*

$$v(\varphi(x_1, x'), x_1, x') \in H^{s'}_{loc}(\mathbf{R}^n_+).$$

If v *and* φ *have compact support, then* $\|v(\varphi(x_1, x'), x_1, x')\|_{H^{s'}} \le C\|\varphi\|_{H^{s'}}$ *with* C *depending only on* s, s', n, *the size of the supports,* $\|v(t, x_1, x')\|_{H^{s+1}}$ *and* $\|\varphi\|_{L^\infty}$.

Proof. We may assume without loss of generality that v and φ have compact support. Let w represent the vector consisting of all derivatives of $\partial_t v$, i.e. $w = (\partial_{x_1}(\partial_t v), \cdots, \partial_{x_n}(\partial_t v))$. Then $w \in H^{s-1}(\mathbf{R} \times \mathbf{R}^n_+) \cap L^\infty(\mathbf{R} \times \mathbf{R}^n_+)$ since $s - 1 > \frac{n}{2}$, and therefore, by Lemma 2.1, $D^\alpha w \in L^{\frac{2(s-1)}{|\alpha|}}(\mathbf{R} \times \mathbf{R}^n_+)$ for $0 \le |\alpha| \le s - 1$ and

$$\|D^\alpha w\|_{L^{2p}} \le C(\|w\|_{L^\infty})^{1-\frac{1}{p}}(\|w\|_{H^{s-1}})^{\frac{1}{p}},$$

where $p = \frac{2(s-1)}{|\alpha|}$ and C is a constant depending only on s and n. Thus $D^\beta \partial_t v \in L^{\frac{2(s-1)}{|\beta|-1}}(\mathbf{R} \times \mathbf{R}^n_+)$ for $1 \le |\beta| \le s$. Since

$$\left(D^\beta v\right)(\varphi(x_1, x'), x_1, x') = \int_{-\infty}^{\varphi(x_1, x')} D^\beta \partial_t v(t, x_1, x')\,dt, \tag{2.1}$$

we apply Minkowski's integral inequality to get

$$\begin{aligned}
&\Big(\int\int_0^\infty \big|\Big(D^\beta v\Big)(\varphi(x_1,x'),x_1,x')\Big|^r\, dx_1dx'\Big)^{1/r}\\
&=\Big(\int\int_0^\infty \Big|\int_{-\infty}^{\varphi(x_1,x')} D^\beta\partial_t v(t,x_1,x')dt\Big|^r\, dx_1dx'\Big)^{1/r}\\
&\le \int_{-\infty}^{\varphi(x_1,x')}\Big(\int_0^\infty\int \Big|D^\beta\partial_t v(t,x_1,x')\Big|^r\, dx_1dx'\Big)^{1/r}dt,
\end{aligned}$$

where $r=\frac{2(s-1)}{|\beta|-1}$, and so we have $\left(D^\beta v\right)(\varphi(x_1,x'),x_1,x')\in L^r(\mathbf{R}^n_+)$, with norm depending only on the size of the supports, $\|v(t,x_1,x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

Next, notice that $D\varphi\in H^{s'-1}(\mathbf{R}^n_+)\cap L^\infty(\mathbf{R}^n_+)$, and therefore $D^\beta\varphi\in L^{\frac{2(s'-1)}{|\beta|-1}}(\mathbf{R}^n_+)$ for $1\le|\beta|\le s'$, by Lemma 2.1. The chain rule and the Leibniz formula imply that, for $1\le|\gamma|\le s'$, $D^\gamma\left(v(\varphi(x_1,x'),x_1,x')\right)$ may be written as a sum of terms of the form

$$\Big(D^{\beta_0}D^m v\Big)(\varphi(x_1,x'),x_1,x')\Big(D^{\beta_1}\varphi(x_1,x')\Big)\cdots\Big(D^{\beta_m}\varphi(x_1,x')\Big),$$

where D^m stands for a derivative of order m with respect to t with $0\le m\le s'$ and $\beta_0+\beta_1+\cdots+\beta_m=\gamma$, $1\le|\beta_k|\le s'$ for $k=0,1,\cdots,m$.

If $m=0$, then $\gamma=\beta_0$, and so $\left(D^{\beta_0}v\right)(\varphi(x_1,x'),x_1,x')\in L^{\frac{2(s-1)}{|\beta_0|-1}}(\mathbf{R}^n_+)$ $\subset L^2(\mathbf{R}^n_+)$ with $\|(D^{\beta_0}v)(\varphi(x_1,x'),x_1,x'))\|_{L^2}\le C$, where C depends only on the size of supports, s, s', n, $\|v(t,x_1,x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$. Therefore, we may assume that $m\ge 1$. The preceding estimates and Hölder's inequality imply that the use of the chain rule was justified, and

$$\big(D^{\beta_0}D^m v\big)(\varphi(x_1,x'),x_1,x')\big(D^{\beta_1}\varphi(x_1,x')\big)\cdots\big(D^{\beta_m}\varphi(x_1,x')\big)\in L^2(\mathbf{R}^n_+)$$

since

$$\frac{|\beta_0|+(m-1)}{2(s-1)}+\frac{(|\beta_1|-1)+\cdots+(|\beta_m|-1)}{2(s'-1)}\le\frac{|\gamma|-1}{2(s'-1)}\le\frac{1}{2}.$$

Therefore $D^\gamma(v(\varphi(x_1,x'),x_1,x'))\in L^2(\mathbf{R}^n_+)$ for $1\le|\gamma|\le s'$. Moreover, by Lemma 2.1, $\|D^\gamma(v(\varphi(x_1,x'),x_1,x'))\|_{L^2}$ is bounded up to an appropriate constant by

$$(\|D\varphi\|_{H^{s'-1}})^{\frac{|\beta_1|-1}{s'-1}}\cdots(\|D\varphi\|_{H^{s'-1}})^{\frac{|\beta_m|-1}{s'-1}}.$$

Since $(|\beta_1|-1)+\cdots+(|\beta_m|-1) \le |\gamma|-m \le s'-m \le s'-1$, the required estimate holds. $\square$

In the proof of conormal regularity of a solution of the equation $P_2u = f(Du)$, where f is smooth of its arguments, we will encounter situations in which regularity of φ with respect to v is greater than the case considered above.

Corollary 2.3. *Let $v(t,x_1,x') \in H^{s+1}_{loc}(\mathbf{R}\times\mathbf{R}^n_+)$ for $s > \frac{n}{2}$. Suppose that $\varphi(x_1,x') \in H^{s+1}_{loc}(\mathbf{R}^n_+)$ and that $D\varphi(x_1,x') \in L^\infty(\mathbf{R}^n_+)$. Then*

$$v(\varphi(x_1,x'),x_1,x') \in H^s_{loc}(\mathbf{R}^n_+).$$

If v and φ have compact support, then $\|v(\varphi(x_1,x'),x_1,x')\|_{H^{s'}} \le C\|\varphi\|_{H^{s'}}$ with C depending only on s, n, the size of the supports, $\|v(t,x_1,x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

Proof. We may assume without loss of generality that v and φ have compact support. Since $s > \frac{n}{2}$, $\partial_t v \in H^s(\mathbf{R}\times\mathbf{R}^n_+) \cap L^\infty(\mathbf{R}\times\mathbf{R}^n_+)$. Therefore, by Lemma 2.1, $D^\beta\partial_t v \in L^{\frac{2s}{|\beta|}}(\mathbf{R}\times\mathbf{R}^n_+)$ for $1 \le |\beta| \le s$. As we have seen in the proof of the previous lemma, it follows from Minkowski's integral inequality and (2.1) that $\left(D^\beta v\right)(\varphi(x_1,x'),x_1,x') \in L^{\frac{2s}{|\beta|}}(\mathbf{R}^n_+)$, with norm depending only on s, n, the size of the supports, $\|v(t,x_1,x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

We also notice that $D\varphi \in H^{s'-1}(\mathbf{R}^n_+)\cap L^\infty(\mathbf{R}^n_+)$, and therefore $D^\beta\varphi \in L^{\frac{2s}{|\beta|-1}}(\mathbf{R}^n_+)$ for $1 \le |\beta| \le s'$, by Lemma 2.1. The chain rule and the Leibniz formula imply that, for $1 \le |\gamma| \le s'$, $D^\gamma\big(v(\varphi(x_1,x'),x_1,x')\big)$ may be written as a sum of terms of the form

$$\big(D^{\beta_0}D^m v\big)(\varphi(x_1,x'),x_1,x')\big(D^{\beta_1}\varphi(x_1,x')\big)\cdots\big(D^{\beta_m}\varphi(x_1,x')\big),$$

where D^m stands for a derivative of order m with respect to t with $0 \le m \le s$ and $\beta_0+\beta_1+\cdots+\beta_m = \gamma$, $1 \le |\beta_k| \le s$ for $k = 0,1,\cdots,m$.

If $m = 0$, then $\gamma = \beta_0$, and so $\left(D^{\beta_0}v\right)(\varphi(x_1,x'),x_1,x') \in L^{\frac{2s}{|\beta_0|}}(\mathbf{R}^n_+) \subset L^2(\mathbf{R}^n_+)$ with $\|(D^{\beta_0}v)(\varphi(x_1,x'),x_1,x'))\|_{L^2} \le C$, where C depends only on the size of supports, s, n, $\|v(t,x_1,x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$. Therefore, we may assume that $m \ge 1$. The preceding estimates and Hölder's inequality imply that the use of the chain rule was justified, and

$$\big(D^{\beta_0}D^m v\big)(\varphi(x_1,x'),x_1,x')\big(D^{\beta_1}\varphi(x_1,x')\big)\cdots\big(D^{\beta_m}\varphi(x_1,x')\big) \in L^2(\mathbf{R}^n_+)$$

since
$$\frac{|\beta_0|+m}{2s}+\frac{(|\beta_1|-1)+\cdots+(|\beta_m|-1)}{2s}\le\frac{|\gamma|}{2s}\le\frac{1}{2}.$$
Therefore $D^\gamma(v(\varphi(x_1,x'),x_1,x'))\in L^2(\mathbf{R}^n_+)$ for $1\le|\gamma|\le s$. Moreover, by Lemma 2.1, $\|D^\gamma(v(\varphi(x_1,x'),x_1,x'))\|_{L^2}$ is bounded up to an appropriate constant by
$$(\|D\varphi\|_{H^s})^{\frac{|\beta_1|-1}{s}}\cdots(\|D\varphi\|_{H^s})^{\frac{|\beta_m|-1}{s}}.$$
Since $(|\beta_1|-1)+\cdots+(|\beta_m|-1)\le s-1$, the required estimate holds. □

The defining function for the characteristic hypersurface $\{t=\varphi(x_1,x')\}$ associated with the quasilinear equation (1.2) satisfies (1.3). Therefore, from (1.3), φ_{x_1} may be expressed locally as a smooth function of $x_1,x',u(\varphi(x_1,x'),x_1,x')$, $Du(\varphi(x_1,x'),x_1,x')$ and $\tilde{D}\varphi(t,x')$, where $\tilde{D}\varphi$ is the x' gradient of φ. Such a function will be denoted by $f(v(\varphi(x_1,x'),x_1,x'),\tilde{D}\varphi(x_1,x'))$, with v representing the vector (t,x_1,x',u,Du). From now on, we use D as total derivative and $\tilde{D}$ as x' derivative. Then our main theorem:

THEOREM 2.4. *Let $v(t,x_1,x')\in H^{s+1}_{loc}(\mathbf{R}\times\mathbf{R}^n_+))$ for $s>\frac{n}{2}+1$, and assume that $\varphi(x_1,x')\in H^s_{loc}(\mathbf{R}^n_+)$ and $D^2\varphi(x_1,x')\in L^\infty_{loc}(\mathbf{R}^n_+)$. Let f be a smooth function of its arguments, and suppose that*
$$\varphi_{x_1}(x_1,x')=f(v(\varphi(x_1,x'),x_1,x'),\tilde{D}\varphi(x_1,x')).$$
If $\varphi(0,x')\in H^s_{loc}(\mathbf{R}^{n-1})$, then $\varphi(x_1,x')\in H^s_{loc}(\mathbf{R}^n_+))$.

Proof. It can be assumed that the functions in question all have compact support in x'. Let $\varphi^{(s)}$ denote the vector of all x' derivatives of φ up to order s. Under the assumption that φ is smooth, we will establish an *a priori* estimate on the energy $E(x_1)=\left(\int|\varphi^{(s)}(x_1,x')|^2dx'\right)^{\frac{1}{2}}$. Standard arguments then allow the smoothness assumption to be dropped.

The chain rule and the Leibniz formula imply that there are smooth functions F and f_α for $|\alpha_1|+\cdots+|\alpha_k|+|\alpha_{k+1}|+\cdots+|\alpha_m|\le s$, $k\ge 1$, which are then evaluated at $\big(v(\varphi(x_1,x'),x_1,x'),\tilde{D}\varphi(x_1,x')\big)$, such that
$$\partial_{x_1}\varphi^{(s)}=F(\varphi,\tilde{D}\varphi)\tilde{D}\varphi^{(s)}(x_1,x')+\sum_\alpha f_\alpha(\varphi,\tilde{D}\varphi)\tilde{D}^{\alpha_1}(v(\varphi,x_1,x'))\cdots$$
$$\tilde{D}^{\alpha_k}(v(\varphi,x_1,x'))\tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi)\cdots\tilde{D}^{\alpha_m}(\tilde{D}\varphi).$$
By differentiating $E(x_1)^2$ with respect to x_1, the energy satisfies

$$E(x_1)\partial_{x_1}E(x_1) = \int \varphi^{(s)}(x_1,x')\partial_{x_1}\varphi^{(s)}(x_1,x')dx',$$

and by integration by parts,

$$\begin{aligned}\int \varphi^{(s)}F(v,\tilde{D}\varphi)\big(\tilde{D}\varphi^{(s)}\big)dx' &= \frac{1}{2}\int F(v,\tilde{D}\varphi)\tilde{D}\big((\varphi^{(s)})^2\big)dx' \\ &= -\frac{1}{2}\int \tilde{D}F(v,\tilde{D}\varphi)\big(\varphi^{(s)}\big)^2dx'.\end{aligned}$$

We note, by the chain rule, that $\tilde{D}F(v,\tilde{D}\varphi) = \sum_\sigma F_\sigma(\tilde{D}v) + G_\sigma(\tilde{D}^2\varphi)$, where F_σ and G_σ are smooth functions of their arguments v, $\tilde{D}\varphi$.

Since $v(t,x_1,x') \in H^{s+1}(\mathbf{R}\times\mathbf{R}^n_+))$ and $s > \frac{n}{2}+1$, the Sobolev imbedding theorem and Lemma 2.2 imply that $v(\varphi,x_1,x') \in L^\infty_{loc}(\mathbf{R}^n_+)$, $D\big(v(\varphi,x_1,x')\big) \in L^\infty_{loc}(\mathbf{R}^n_+)$, and since $D^2\varphi(x_1,x') \in L^\infty_{loc}(\mathbf{R}^n_+)$, $\tilde{D}(F(v(t,\varphi,x'),\tilde{D}\varphi(t,x'))) \in L^\infty_{loc}(\mathbf{R}^n_+)$ and $f_\alpha(v(t,\varphi,x'),\tilde{D}\varphi(t,x')) \in L^\infty_{loc}(\mathbf{R}^n_+)$.

By Lemma 2.2, $v(\varphi(x_1,x'),x_1,x') \in H^s(\mathbf{R}^n_+))$ with $\|v(\varphi(x_1,x'),x_1,x')\|_{H^s} \le C(x_1)E(x_1)$. Therefore

$$\tilde{D}(v(\varphi(x_1,x'),x_1,x')) \in H^{s-1}(\mathbf{R}^n_+)) \cap L^\infty_{loc}(\mathbf{R}^n_+)$$

with similar bounds on the $H^{s-1}(\mathbf{R}^n_+)$ norm. Hence from Lemma 2.1 ,

$$\tilde{D}^{\alpha_j}\left(v(t,\varphi(t,x'),x')\right) \in L^{2p_j}(\mathbf{R}^n_+), \quad \text{where} \quad p_j = \frac{s-1}{|\alpha_j|-1}$$

and

$$\begin{aligned}(2.2)\qquad &\Big\|\tilde{D}^{\alpha_j}\Big(v(\varphi(x_1,x'),x_1,x')\Big)\Big\|_{L^{2p_j}} \\ &\le C\|\tilde{D}v\|_{L^\infty}^{1-\frac{1}{p_j}}\|\tilde{D}v\|_{H^{s-1}}^{\frac{1}{p_j}} \le C(x_1)\Big(E(x_1)\Big)^{\frac{1}{p_j}},\end{aligned}$$

for $1 \le |\alpha_j| \le s$, $j = 1,\cdots,k$. Similarly,

$$(D\varphi) \in H^{s-1}(\mathbf{R}^n_+)) \cap L^\infty_{loc}(\mathbf{R}^n_+),$$

with $\|D\varphi\|_{H^{s-1}} \le E(x_1)$ and $\|D\varphi\|_{L^\infty} \le C(x_1)$. Thus, again from Lemma 2.1,

$$\tilde{D}^{\alpha_j}\left(\tilde{D}\varphi(x_1,x')\right) \in L^{2q_j}(\mathbf{R}^n_+), \quad \text{where} \quad q_j = \frac{s-1}{|\alpha_j|},$$

and

$$(2.3)\quad \|\tilde{D}^{\alpha_j}(\tilde{D}\varphi)\|_{L^{2q_j}} \le C\|D\varphi\|_{L^\infty}^{1-\frac{1}{q_j}}\|D\varphi\|_{H^{s-1}}^{\frac{1}{q_j}} \le C(x_1)\left(E(x_1)\right)^{\frac{1}{q_j}},$$

for $0 \le |\alpha_j| \le s-1$, $j = k+1, \cdots, m$. Therefore, by Hölder's inequality,

$$\tilde{D}^{\alpha_1}\left(v(t,\varphi(t,x'),x')\right)\cdots\tilde{D}^{\alpha_k}\left(v(t,\varphi(t,x'),x')\right)\tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi)\\ \cdots\tilde{D}^{\alpha_m}(\tilde{D}\varphi)\in L^2(\mathbf{R}^n_+),$$

since $((|\alpha_1|-1)+\cdots+(|\alpha_k|-1)+|\alpha_{k+1}|+\cdots+|\alpha_m|)/2(s-1)\le(s-k)/2(s-1)\le 1/2$. From (2.3) and (2.3),

$$\left\|(\tilde{D}^{\alpha_1}v)\cdots(\tilde{D}^{\alpha_k}v)\tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi)\cdots\tilde{D}^{\alpha_m}(\tilde{D}\varphi)\right\|_{L^2}\\ \le C(x_1)\left(E(x_1)\right)^{\frac{s-k}{s-1}}\le C(x_1)E(x_1). \tag{2.4}$$

By Minkowski's inequality, Schwarz's inequality and (2.4),

$$|E(x_1)\partial_{x_1}E(x_1)|\\ \le C\|\tilde{D}F\|_{L^\infty}\int\left(\varphi^{(s)}\right)^2dx'+C\left(\int\left(\varphi^{(s)}\right)^2dx'\right)^{\frac{1}{2}}\cdot\\ \sum_\alpha\|f_\alpha(\tilde{D}^{\alpha_1}v(t,\varphi,x'))\cdots(\tilde{D}^{\alpha_k}v(t,\varphi,x'))\tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi)\\ \cdots\tilde{D}^{\alpha_m}(\tilde{D}\varphi)\|_{L^2}\\ \le CE(x_1)\left(\|\tilde{D}F\|_{L^\infty}E(x_1)+\|f_\alpha\|_{L^\infty}C(x_1)E(x_1)\right),$$

and so $|\partial_{x_1}E(x_1)|\le C(x_1)E(x_1)+C(x_1)$. Therefore, by Gronwall's inequality, $E(x_1)$ is finite for all time since by assumption it is finite at $x_1=0$. □

We will also encounter first order equation like those in Corollary 2.5 which are linear, but with coefficients of finite regularity.

COROLLARY 2.5. *Let $\psi(x_1,x')$ satisfy the equation*

$$\psi_{x_1}=F_1(x_1,x')\tilde{D}\psi+F_2(x_1,x')\psi+F_3(x_1,x'), \tag{2.5}$$

where $F_1(x_1,x')\in H^{s-1}_{loc}(\mathbf{R}^n_+)$ and $F_2(x_1,x'),F_3(x_1,x')\in H^{s-2}_{loc}(\mathbf{R}^n_+)$, $s>\frac{n}{2}+2$. If $\psi(0,x')\in H^{s-2}_{loc}(\mathbf{R}^n)$, then $\psi(x_1,x')\in H^{s-2}_{loc}(\mathbf{R}^n_+)$.

Proof. It can be assumed that the functions in question all have support in x'. Let $\psi^{(s-2)}$ denote the vector of all x' derivatives of ψ up to order $s-2$. As we see in the proof of Theorem 2.4, we will establish *a priori* estimates on the energy $E(x_1)=\left(\int|\psi^{(s-2)}(x_1,x')|^2dx'\right)^{\frac{1}{2}}$.

The chain rule and the Leibniz formula imply that for $|\alpha_1| + |\alpha_2| \le s-1, 1 \le |\alpha_2| \le s-2$ and $|\beta_1| + |\beta_2| \le s-2$,

$$\begin{aligned}
&\partial_{x_1} \psi^{(s-2)} \\
&= F_1(x_1, x')\ \tilde{D}\psi^{(s-2)}(x_1, x') + \sum_\alpha \left(\tilde{D}^{\alpha_1} F_1(x_1, x')\right)\left(\tilde{D}^{\alpha_2}\psi(x_1, x')\right) \\
&\quad + \sum_\beta \left(\tilde{D}^{\beta_1} F_2(x_1, x')\right)\left(\tilde{D}^{\beta_2}\psi(x_1, x')\right) + \tilde{D}^{s-2} F_3(x_1, x').
\end{aligned}$$

The energy satisfies $E(x_1)\partial_{x_1} E(x_1) = \int \psi^{(s-2)}(x_1, x')\partial_{x_1}\psi^{(s-2)}(x_1, x')dx'$, and by integration by parts,

$$\int \psi^{(s-2)} F_1(x_1, x')\big(\tilde{D}\psi^{(s-2)}\big)dx' = -\frac{1}{2}\int \tilde{D}F_1(x_1, x')\big(\psi^{(s-2)}\big)^2 dx'.$$

Since $s > \frac{n}{2}+2$, the Sobolev imbedding theorem implies that $\tilde{D}F_1(x_1, x') \in L^\infty_{loc}(\mathbf{R} \times \mathbf{R}^n_+)$.

Since $\tilde{D}F_1(x_1, x') \in H^{s-2}(\mathbf{R}^n_+) \cap L^\infty_{loc}(\mathbf{R}^n_+)$, by Lemma 2.1 ,

$$\tilde{D}^{\alpha_1} F_1(x_1, x') \in L^{2p_1}(\mathbf{R}^{n-1}), \quad \text{where} \quad p_1 = \frac{s-2}{|\alpha_1| - 1}$$

and

$$\left\|\tilde{D}^{\alpha_1} F_1(t, x')\right\|_{L^{2p_1}} \le C\|\tilde{D}F_1\|_{L^\infty}^{1-\frac{1}{p_1}}\|\tilde{D}F_1\|_{H^{s-2}}^{\frac{1}{p_1}} \le C(x_1). \tag{2.6}$$

for $1 \le |\alpha_1| \le s-1$. Similarly,

$$\psi \in H^{s-2}(\mathbf{R}^n_+) \cap L^\infty_{loc}(\mathbf{R}^n_+),$$

with $\|\psi\|_{H^{s-2}} \le E(t)$ and $\|\psi\|_{L^\infty} \le C(t)$. Thus, again from Lemma 2.1,

$$\tilde{D}^{\alpha_2}\psi(x_1, x') \in L^{2p_2}(\mathbf{R}^{n-1}), \quad \text{where} \quad p_2 = \frac{s-2}{|\alpha_2|},$$

and

$$\|\tilde{D}^{\alpha_2}(\psi)\|_{L^{2p_2}} \le C\|\psi\|_{L^\infty}^{1-\frac{1}{p_2}}\|\psi\|_{H^{s-2}}^{\frac{1}{p_2}} \le C(x_1)\left(E(x_1)\right)^{\frac{1}{p_2}}, \tag{2.7}$$

for $1 \le |\alpha_2| \le s-2$. Therefore, by Hölder's inequality,

$$\left(\tilde{D}^{\alpha_1} F_1(x_1, x')\right)\left(\tilde{D}^{\alpha_2}\psi\right) \in L^2(\mathbf{R}^n_+)),$$

since $((|\alpha_1|-1)+|\alpha_2|)/2(s-2) \le 1/2$. For $|\alpha_1| = 0$, $\|F_1(x_1, x')\tilde{D}\psi(x_1, x')\|_{L^2} \le C\|F_1\|_{L^\infty} E(x_1)$. Therefore, from (2.6) and (2.7),

$$\left\| \left(\tilde{D}^{\alpha_1} F_1(t, x') \right) \left(\tilde{D}^{\alpha_2} \psi(t, x') \right) \right\|_{L^2} \leq C(x_1) E(x_1). \tag{2.8}$$

Since $F_2(x_1, x') \in H^{s-2}(\mathbf{R}^n_+) \cap L^\infty_{loc}(\mathbf{R}^n_+)$, from Lemma 2.1,

$$\tilde{D}^{\beta_1} \left(F_2(x_1, x') \right) \in L^{2q_1}(\mathbf{R}^{n-1}), \quad \text{where} \quad q_1 = \frac{s-2}{|\beta_1|}$$

and

$$\left\| \tilde{D}^{\beta_1} \left(F_2(x_1, x') \right) \right\|_{L^{2q_1}} \leq C \|F_2\|_{L^\infty}^{1-\frac{1}{p_j}} \|F_2\|_{H^{s-2}}^{\frac{1}{q_1}} \leq C(x_1), \tag{2.9}$$

for $0 \leq |\beta_1| \leq s-2$. Also, again from Lemma 2.1,

$$\tilde{D}^{\beta_2} \psi(x_1, x')) \in L^{2q_2}(\mathbf{R}^{n-1}), \quad \text{where} \quad q_2 = \frac{s-2}{|\beta_2|},$$

and

$$\|\tilde{D}^{\beta_2} \psi\|_{L^{2q_2}} \leq C \|\psi\|_{L^\infty}^{1-\frac{1}{q_2}} \|\psi\|_{H^{s-2}}^{\frac{1}{q_2}} \leq C(x_1) \left(E(x_1) \right)^{\frac{1}{q_2}}, \tag{2.10}$$

for $0 \leq |\beta_2| \leq s-2$. Therefore, by Hölder's inequality,

$$\tilde{D}^{\beta_1} \left(F_2(x_1, x') \right) \tilde{D}^{\beta_2} \psi \in L^2(\mathbf{R}^{n-1}),$$

since $(|\beta_1| + |\beta_2|)/2(s-2) \leq 1/2$. From (2.9) and (2.10),

$$\left\| \tilde{D}^{\beta_1} \left(F_2(t, x') \right) \tilde{D}^{\beta_2} \psi(t, x') \right\|_{L^2} \leq C(t) E(t). \tag{2.11}$$

Therefore, by Minkowski's inequality, Schwarz's inequality, (2.9) and (2.11),

$$\begin{aligned}
&|E(x_1) \partial_{x_1} E(x_1)| \\
&\leq C \|\tilde{D} F_1\|_{L^\infty} \int \left(\varphi^{(s-2)} \right)^2 dx' + C \Big(\int \left(\varphi^{(s-2)} \right)^2 dx' \Big)^{\frac{1}{2}} + \\
&\quad \Big\{ \sum_\alpha \|(\tilde{D}^{\alpha_1} F_1)(\tilde{D}^{\alpha_2} \psi)\|_{L^2} + \sum_\beta \|(\tilde{D}^{\beta_1} F_2)(\tilde{D}^{\beta_2} \psi)\|_{L^2} \\
&\quad + \|\tilde{D}^{s-2} F_3\|_{L^2} \Big\} \\
&\leq C E(x_1) \left(\|\tilde{D} F_1\|_{L^\infty} E(x_1) + C(x_1) E(x_1) + C(x_1) \right),
\end{aligned}$$

and so $|\partial_{x_1} E(x_1)| \leq C(x_1) E(x_1) + C(x_1)$. Therefore, by Gronwall's inequality, $E(t)$ is finite for all time since by assumption it is finite at $x_1 = 0$. $\square$

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